Exact results for the nonlinear diffusion equations delta u/ delta t= delta / delta x(u$^{-4/3}$ delta u/ delta x) and delta u/ delta t- delta / delta x(u$^{-2/3}$ delta u/ delta x)

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Exact results for the nonlinear diffusion equations $\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right)$ and $\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-2/3}\frac{\partial u}{\partial x}\right)$

J R King

Department of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD, UK

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Abstract. We exploit local and non-local symmetries of the two equations of the title and of some related equations. Some new exact solutions are derived and some specific boundary value problems of physical significance are considered.

1. Introduction

The nonlinear diffusion equation with power law diffusivity

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-m}\frac{\partial u}{\partial x}\right) \tag{1.1}$$

arises in a large number of physical contexts. This paper is largely concerned with two particular instances of (1.1) both belonging to the 'fast' diffusion class ($m>0$), namely the cases $m=\frac{4}{3}$ and $m=\frac{2}{3}$.

We shall make use of Lie group methods for partial differential equations; the necessary background to these is given in, for example, Bluman and Cole [1]. We shall apply the group properties determined by Akhatov *et al* [2] for (1.1) and for some related equations; their results show the respect in which the values $m=\frac{4}{3}$ and $m=\frac{2}{3}$ are special, and therefore worthy of separate consideration. In fact, it has been known for some time (Ovsiannikov [3]) that the invariance group of equation (1.1) with $m=\frac{4}{3}$ has one more parameter than that for other values of $m\neq 0$, so that in the former case there are more self-similar forms of solution. It has also been shown (Munier *et al* [4]) that there is a non-local transformation between two equations of the form (1.1) with $m=m_1$ and $m=m_2$, provided that $m_1+m_2=2$. Hence the two equations we are discussing can be mapped into one another.

Writing $u=\partial v/\partial x$, equation (1.1) with $m\neq 1$ becomes

$$\frac{\partial v}{\partial t}=\frac{1}{1-m}\frac{\partial}{\partial x}\left(\left(\frac{\partial v}{\partial x}\right)^{1-m}\right). \tag{1.2}$$

Even though the values of m which we are considering are very special, applications are known for each. A slightly generalized version of equation (1.1) with $m=\frac{4}{3}$ is used by Henninger *et al* [5] to describe heat conduction in silicon, while (1.2) with $m=\frac{2}{3}$ models heat transfer in superfluid helium (Dresner [6, 7]). Furthermore, equation (1.1) with m lying in the appropriate range arises in many other applications. For example, models of dopant diffusion in silicon give (King [8])

$$m = \frac{M-1}{M+1}$$

where M is the number of atoms in a dopant cluster. The choice $M=5$ gives $m=\frac{2}{3}$. The aim of this paper is to exploit the special group properties for $m=\frac{4}{3}$ and $m=\frac{2}{3}$ to derive new exact solutions and to analyse some specific boundary value problems, including some on finite domains. We start by outlining the relevant Lie group properties.

2. Group properties

In table 1 we summarize some of the relevant relationships which have been established elsewhere. This table follows from table 1 of King [9], though the notation is different.

Table 1.

(2.1) $u_t = (u^{-4/3}u_x)_x$	$\overset{u=v_x}{\longleftrightarrow}$	(2.2) $v_t = v_x^{-4/3}v_{xx}$	$\overset{v=w_x}{\longleftrightarrow}$	(2.3) $w_t = -3w_{xx}^{-1/3}$
		↕		⋮
(2.4) $U_T = (U^{-2/3}U_X)_X$	$\overset{U=V_X}{\longleftrightarrow}$	(2.5) $V_T = V_X^{-2/3}V_{XX}$	$\overset{V=W_X}{\longleftrightarrow}$	(2.6) $W_T = 3W_{XX}^{1/3}$
				↕
(2.7) $U^*_{T^*} = (U^{*-2/3}U^*_{X^*})_{X^*}$	$\overset{U^*=V^*_{X^*}}{\longleftrightarrow}$	(2.8) $V^*_{T^*} = V^{*-2/3}_{X^*}V^*_{X^*X^*}$	$\overset{V^*=W^*_{X^*}}{\longleftrightarrow}$	(2.9) $W^*_{T^*} = 3W^{*1/3}_{X^*X^*}$
		↕		⋮
(2.10) $u^*_{t^*} = (u^{*-4/3}u^*_{x^*})_{x^*}$	$\overset{u^*=v^*_{x^*}}{\longleftrightarrow}$	(2.11) $v^*_{t^*} = v^{*-4/3}_{x^*}v^*_{x^*x^*}$	$\overset{v^*=w^*_{x^*}}{\longleftrightarrow}$	(2.12) $w^*_{t^*} = -3w^{*-1/3}_{x^*x^*}$

In table 1 the vertical solid lines represent hodograph transformations with

$$V = x \qquad X = v \qquad T = t \tag{2.13}$$

$$V^* = x^* \qquad X^* = v^* \qquad T^* = t^* \tag{2.14}$$

and

$$W^* = X \qquad X^* = W \qquad T^* = T \qquad \text{with } V^* = 1/V,\ U^* = -U/V^3. \tag{2.15}$$

The vertical dotted lines denote Legendre transformations, so that

$$w + W = xX \qquad w^* + W^* = x^*X^*. \tag{2.16}$$

There are point transformations between equations (2.1) and (2.10) and between (2.3) and (2.12) given by

$$u^* = -x^3u \qquad w^* = -w/x \qquad x^* = 1/x \qquad t^* = t \tag{2.17}$$

and the additional relationships

$$U = 1/u \qquad U^* = 1/u^* \tag{2.18}$$

and

$$v^* = xv - w \qquad v = x^* v^* - w^* \tag{2.19}$$

also hold.

With regard to (2.17) it should be pointed out that certain local relationships between variables appearing in tables 1 and 2 of King [9] were not noted there, namely:

$$\psi = -z/v \qquad q = -p/u \qquad \varphi = -w/y.$$

These provide point transformations between equations (3) and (14), (8) and (17), and (11) and (22) of table 1, and equations (3) and (10) of table 2 in King [9]. In addition, in table 2 the relation $\xi = 1/y$ was misprinted as $\xi = 1/u$.

In what follows we shall make use of results of Akhatov *et al* [10, 2]. Having introduced the concept of non-local (or quasi-local) symmetries in [10], they gave the infinitesimal forms relevant to (2.1)–(2.6) in [2]. Here we shall use ˆ to denote the infinitesimals, and * to denote the corresponding global forms of the symmetries, so that we write

$$u^*(x, t, u, v, w; \varepsilon) \sim u + \varepsilon \hat{u}(x, t, u, v, w)$$

as $\varepsilon \to 0$, and similarly for the other variables. Thus the equation in unstarred variables maps in each case into the same equation in starred variables. The infinitesimal forms of the quasi-local symmetries for (2.1)–(2.6) given by Akhatov *et al* [2] may then be written as

$$\begin{aligned} &\hat{u} = a_6 u + 3a_7 xu \qquad \hat{v} = a_4 + (a_3 + a_6)v - a_7(w - xv) \\ &\hat{w} = a_5 + a_4 x + (2a_3 + a_6)w - a_7 xw \\ &\hat{x} = a_2 + a_3 x - a_7 x^2 \qquad \hat{t} = a_1 + (2a_3 + \tfrac{4}{3}a_6)t \end{aligned} \tag{2.20}$$

for (2.1)–(2.3), and as

$$\begin{aligned} &\hat{U} = A_6 U - 3A_7 UV \qquad \hat{V} = A_4 + (A_3 + A_6)V - A_7 V^2 \\ &\hat{W} = A_5 + A_4 X + (2A_3 + A_6)W \\ &\hat{X} = A_2 + A_3 X + A_7 W \qquad \hat{T} = A_1 + (2A_3 + \tfrac{2}{3}A_6)T \end{aligned} \tag{2.21}$$

for (2.4)–(2.6). In (2.20) and (2.21) the a_i and A_i are arbitrary constants. The parameters a_7 and A_7 give the symmetries for which there are no corresponding results for (1.1) when m is not $\frac{2}{3}$ or $\frac{4}{3}$.

Setting $a_7 = 1$ and $a_i = 0$ for $i = 1$–6 in (2.20) we may deduce the global forms

$$\begin{aligned} &u^* = (1 + \varepsilon x)^3 u \qquad v^* = (1 + \varepsilon x)v - \varepsilon w \\ &w^* = w/(1 + \varepsilon x) \qquad x^* = x/(1 + \varepsilon x) \qquad t^* = t. \end{aligned} \tag{2.22}$$

We note that this is a local symmetry for u and w, whereas for v it is non-local because v^* depends on $w = \int^x v \, dx$. The global forms for u^* and x^* were noted in King [11].

The discrete transformation given by (2.17) and (2.19) may be derived as a special case of (2.22) combined with appropriate translations, rescalings and reflections of the variables.

Setting $A_7 = 1$ and $A_i = 0$ for $i = 1$–6 in (2.21) leads to

$$\begin{aligned} &U^* = U/(1 + \varepsilon V)^3 \qquad V^* = V/(1 + \varepsilon V) \\ &W^* = W \qquad X^* = X + \varepsilon W \qquad T^* = T. \end{aligned} \tag{2.23}$$

This is a local symmetry for W, but it is non-local for U and V since X^* depends on W. We note that the symmetries (2.22) and (2.23) map into one another under the transformations between (2.1)-(2.3) and (2.4)-(2.6) indicated in table 1. The transformation (2.23) may easily be generalized to give

$$U^* = U/(\alpha_2 V+\beta_2)^3 \qquad V^* = (\alpha_1 V+\beta_1)/(\alpha_2 V+\beta_2)$$
$$W^* = \alpha_1 W+\beta_1 X \qquad X^* = \alpha_2 W+\beta_2 X \qquad T^* = T \tag{2.24}$$

where α_1, α_2, β_1, β_2 are constants such that

$$\alpha_1\beta_2-\alpha_2\beta_1 = 1.$$

Since (2.4)-(2.6) are unchanged when X is replaced by $-X$ and V by $-V$, this condition can be generalized slightly to

$$|\alpha_1\beta_2-\alpha_2\beta_1| = 1.$$

The choice $\alpha_2=\beta_1=1$, $\alpha_1=\beta_2=0$ gives the discrete transformation (2.15) as a special case of (2.24).

The transformation (2.22) can be generalized in a similar fashion to give, in particular,

$$u^* = (\alpha_2 x+\beta_2)^3 u \qquad x^* = (\alpha_1 x+\beta_1)/(\alpha_2 x+\beta_2) \qquad t^* = t$$
$$\text{with } |\alpha_1\beta_2-\alpha_2\beta_1| = 1. \tag{2.25}$$

We note that the symmetries for u, w and W are all local (see (2.20) and (2.21)) so that the equations (2.1), (2.3) and (2.6) are the forms which will be simplest to deal with.

3. Some exact solutions

3.1. Known solutions

The purpose of this section is to briefly show how well known exact solutions to (2.1) and (2.4) can be used to generate new exact solutions to (2.1)-(2.6). Applications of some of these new solutions will be given in section 4.

We start by listing some well known exact solutions to (1.1) (see, for example, Hill [12]) and the corresponding forms for

$$\frac{\partial v}{\partial t} = \left(\frac{\partial v}{\partial x}\right)^{-m} \frac{\partial^2 v}{\partial x^2} \tag{3.1}$$

and

$$\frac{\partial w}{\partial t} = \frac{1}{1-m}\left(\frac{\partial^2 w}{\partial x^2}\right)^{1-m} \tag{3.2}$$

where

$$u = \frac{\partial v}{\partial x} \qquad v = \frac{\partial w}{\partial x}.$$

In what follows the μ_i and ν_i are arbitrary constants and the functions $f_i(\eta)$, $g_i(\eta)$ and $h_i(\eta)$ are related for each i by

$$f_i = \frac{\mathrm{d}g_i}{\mathrm{d}\eta} \qquad g_i = \frac{\mathrm{d}h_i}{\mathrm{d}\eta}. \tag{3.3}$$

The expression

$$\frac{\partial w}{\partial t} = \frac{1}{1-m} u^{1-m}$$

leads to a further relationship.

(1) Travelling wave solutions

$$u = f_1(\eta) \qquad \eta = x - qt. \tag{3.4}$$

Here q is a constant and the solution to (1.1) is given by

$$-q\eta = \int_{\mu_1}^{f_1} \frac{f^{-m}}{f - \nu_1}\,\mathrm{d}f. \tag{3.5}$$

For both $m = \frac{4}{3}$ and $m = \frac{2}{3}$ the integral in (3.5) can be calculated explicitly.

The solutions

$$v = q\nu_1 t + g_1(\eta) \tag{3.6}$$

and

$$w = q\nu_1 tx - \tfrac{1}{2}q^2\nu_1 t^2 + h_1(\eta) \tag{3.7}$$

can then be evaluated from (3.3);

$$g_1 = \nu_1\eta - \frac{1}{(1-m)q} f_1^{1-m}$$

also holds.

(2) Instantaneous source solutions

$$u = t^{-1/(2-m)} f_2(\eta) \qquad \eta = x/t^{1/(2-m)}. \tag{3.8}$$

f_2 satisfies

$$\nu_2 - \frac{1}{2-m}\eta f_2 = f_2^{-m}\frac{\mathrm{d}f_2}{\mathrm{d}\eta}$$

where ν_2 is an arbitrary constant. For $\nu_2 = 0$ we obtain the closed form solution

$$f_2 = \left(\frac{m}{2(2-m)}(\mu_2^2 + \eta^2)\right)^{-1/m}. \tag{3.9}$$

The corresponding solutions to (3.1) and (3.2) are

$$v = \nu_2 \ln t + g_2(\eta) \tag{3.10}$$

and

$$w = \nu_2 x \ln t + t^{1/(2-m)} h_2(\eta) \tag{3.11}$$

with

$$h_2 = \eta g_2 + \frac{2-m}{1-m} f_2^{1-m} - (2-m)\nu_2\eta.$$

(3) Dipole solutions

$$u = t^{-1/(1-m)} f_3(\eta) \qquad \eta = x/t^{1/(2(1-m))}. \tag{3.12}$$

We then have

$$\nu_3 - \frac{1}{2(1-m)}\eta^2 f_3 = \eta f_3^{-m}\frac{\mathrm{d}f_3}{\mathrm{d}\eta} - \frac{1}{1-m} f_3^{1-m}.$$

If $\nu_3 = 0$ then

$$f_3 = \left(\frac{m}{2(2-m)}\eta^{-m/(1-m)}\left(\mu_3^{(2-m)/(1-m)} + \eta^{(2-m)/(1-m)}\right)\right)^{-1/m}. \tag{3.13}$$

In this case

$$v = t^{-1/(2(1-m))} g_3(\eta) \tag{3.14}$$

and

$$w = -\nu_3 \ln t + h_3(\eta) \tag{3.15}$$

with

$$g_3 = -2(f_3^{1-m} + (1-m)\nu_3)/\eta.$$

(4) Separable solutions

$$u = t^{1/m} f_4(\eta) \qquad \eta = x. \tag{3.16}$$

f_4 is given by

$$\pm\left(\frac{2}{m(2-m)}\right)^{1/2}\eta = \int_{\mu_4}^{f_4}\frac{f^{-m}}{(f^{2-m} - \nu_4^{2-m})^{1/2}}\,\mathrm{d}f \tag{3.17}$$

(when $m = \frac{4}{3}$ the integral can be calculated explicitly) and

$$v = t^{1/m} g_4(\eta) \tag{3.18}$$

$$w = t^{1/m} h_4(\eta) \tag{3.19}$$

with

$$h_4 = \frac{m}{1-m} f_4^{1-m}$$

and

$$g_4 = \pm\left\{\frac{2m}{2-m}\left(f_4^{2-m} - \nu_4^{2-m}\right)\right\}^{1/2}.$$

(5) Steady state solutions

$$u = f_5(\eta) \qquad \eta = x. \tag{3.20}$$

In this case

$$f_5 = \{(1-m)(\nu_5 + \mu_5\eta)\}^{1/(1-m)} \tag{3.21}$$

so that

$$v = \mu_5 t + \frac{1}{(2-m)\mu_5}\{(1-m)(\nu_5 + \mu_5\eta)\}^{(2-m)/(1-m)} \tag{3.22}$$

and

$$w=\mu_5\eta t+\nu_5 t+\frac{1}{(2-m)(3-2m)\mu_5^2}\{(1-m)(\nu_5+\mu_5\eta)\}^{(3-2m)/(1-m)}. \tag{3.23}$$

We note that under the hodograph transformation

$$V=x \qquad X=v \qquad T=t \tag{3.24}$$

under which equation (1.2) maps into another equation of the same form with m replaced by $2-m$, these solutions transform as follows (in each case $G_i(g_i(\eta))=\eta$):

(*a*) Solution (3.6) maps to

$$V=qT+G_1(X-q\nu_1 T)$$

which is a solution of the same form.

(*b*) Solution (3.10) maps to

$$V=T^{1/(2-m)}G_2(X-\nu_2\ln T).$$

When $\nu_2=0$ this is equivalent to the form (3.18).

(*c*) Solution (3.14) maps to

$$V=T^{1/(2(1-m))}G_3(XT^{1/(2(1-m))})$$

again a solution of the same form.

(*d*) Solution (3.18) maps to

$$V=G_4(X/T^{1/m})$$

corresponding to (3.10) with $\nu_2=0$.

(*e*) Solution (3.22) maps to

$$V=G_5(X-\mu_5 T).$$

Hence the various cases (1)–(5) (with $\nu_2=0$ in (2)) for $m=m_1$ and $m=2-m_1$ map into one another under (3.24).

To derive new exact solutions we will now restrict attention to the cases $m=\frac{4}{3}$ and $m=\frac{2}{3}$, and make use of the symmetries (2.22) and (2.24). The existence of extra symmetries for these two cases means that there are many additional similarity forms. Here we only discuss the cases where the solutions can be written down exactly.

3.2. Further solutions for $m=\frac{4}{3}$

From (2.22) it follows that if

$$u=u_0(x,t) \qquad v=v_0(x,t) \qquad w=w_0(x,t)$$

satisfy equations (2.1)–(2.3), with

$$u_0=\frac{\partial v_0}{\partial x} \qquad v_0=\frac{\partial w_0}{\partial x}$$

then so do

$$u=(1+\varepsilon x)^{-3}u_0\left(\frac{x}{1+\varepsilon x},t\right)$$
$$v=(1+\varepsilon x)^{-1}v_0\left(\frac{x}{1+\varepsilon x},t\right)+\varepsilon w_0\left(\frac{x}{1+\varepsilon x},t\right) \tag{3.25}$$
$$w=(1+\varepsilon x)w_0\left(\frac{x}{1+\varepsilon x},t\right).$$

By using the solutions given in (1)–(5) above to give u_0, v_0 and w_0, we may generate the following further exact solutions:

(1) From (3.4)–(3.7) we obtain

$$u=(1+\varepsilon x)^{-3}f_1(\eta)$$
$$v=q\nu_1 t-\tfrac{1}{2}\varepsilon q^2\nu_1 t^2+(1+\varepsilon x)^{-1}g_1(\eta)+\varepsilon h_1(\eta) \tag{3.26}$$
$$w=q\nu_1 tx-\tfrac{1}{2}(1+\varepsilon x)q^2\nu_1 t^2+(1+\varepsilon x)h_1(\eta)$$

where $\eta=x/(1+\varepsilon x)-qt$.

(2) From (3.8)–(3.11) with $\nu_2=0$ we obtain

$$u=(1+\varepsilon x)^{-3}t^{-3/2}f_2(\eta)$$
$$v=(1+\varepsilon x)^{-1}g_2(\eta)+\varepsilon t^{3/2}h_2(\eta) \tag{3.27}$$
$$w=(1+\varepsilon x)t^{3/2}h_2(\eta)$$

where $\eta=x/(1+\varepsilon x)t^{3/2}$.

(3) From (3.12)–(3.15) with $\nu_3=0$ we obtain

$$u=(1+\varepsilon x)^{-3}t^3f_3(\eta)$$
$$v=(1+\varepsilon x)^{-1}t^{3/2}g_3(\eta)+\varepsilon h_3(\eta) \tag{3.28}$$
$$w=(1+\varepsilon x)h_3(\eta)$$

where $\eta=xt^{3/2}/(1+\varepsilon x)$.

We note that the solutions (3.27) and (3.28) are equivalent to within translations of x, etc; under the discrete transformation (2.17) the instantaneous source and dipole solutions of (2.1) are mapped into one another.

In the cases (4) and (5) the solutions simply map into other solutions of the same form.

3.3. *Further solutions for $m=\frac{2}{3}$*

It follows from (2.24) that if

$$U=U_0(X,T) \qquad V=V_0(X,T) \qquad W=W_0(X,T)$$

satisfy equations (2.4)–(2.6) with

$$U_0=\frac{\partial V_0}{\partial X} \qquad V_0=\frac{\partial W_0}{\partial X}$$

then further solutions are given in implicit form by

$$\alpha_1 W + \beta_1 X = W_0(\alpha_2 W + \beta_2 X, T)$$
$$V = -(\beta_2 V_0(\alpha_2 W + \beta_2 X, T) - \beta_1)/(\alpha_2 V_0(\alpha_2 W + \beta_2 X, T) - \alpha_1)$$
$$U = -U_0(\alpha_2 W + \beta_2 X, T)/(\alpha_2 V_0(\alpha_2 W + \beta_2 X, T) - \alpha_1)^3$$

with $|\alpha_1\beta_2 - \alpha_2\beta_1| = 1$.

Since translating W by a multiple of X has no effect on U, we take $\beta_1 = 0$ if $\alpha_1 \neq 0$ and $\beta_2 = 0$ if $\alpha_1 = 0$. We may therefore without significant loss of generality restrict attention to the cases

$$\alpha_1 = \beta_2 = 1 \qquad \beta_1 = 0 \qquad \alpha_2 = \alpha$$

so that

$$W = W_0(X + \alpha W, T)$$
$$V = V_0(X + \alpha W, T)/(1 - \alpha V_0(X + \alpha W, T)) \tag{3.29}$$
$$U = U_0(X + \alpha W, T)/(1 - \alpha V_0(X + \alpha W, T))^3$$

and

$$\alpha_1 = \beta_2 = 0 \qquad \alpha_2 = \beta_1 = 1$$

when

$$X = W_0(W, T) \qquad V = 1/V_0(W, T) \qquad U = -U_0(W, T)/V_0^3(W, T). \tag{3.30}$$

These correspond to (2.23) and (2.15) respectively.

These transformations generate further solutions as follows:

(1) From (3.4)-(3.7), using (3.29) we get

$$W = q\nu_1 T(X + \alpha W) - \tfrac{1}{2}q^2\nu_1 T^2 + h_1(\eta)$$
$$V = (q\nu_1 T + g_1(\eta))/(1 - \alpha(q\nu_1 T + g_1(\eta))) \tag{3.31}$$
$$U = f_1(\eta)/(1 - \alpha(q\nu_1 T + g_1(\eta)))^3$$

where $\eta = X + \alpha W - qT$, whereas (3.30) gives

$$X = q\nu_1 TW - \tfrac{1}{2}q^2\nu_1 T^2 + h_1(W - qT)$$
$$V = 1/(q\nu_1 T + g_1(W - qT)) \tag{3.32}$$
$$U = -f_1(W - qT)/(q\nu_1 T + g_1(W - qT))^3.$$

(2) When $\nu_2 = 0$, the appropriate forms in (3.8)-(3.11) are

$$f_2(\eta) = 8(\mu_2^2 + \eta^2)^{-3/2}$$
$$g_2(\eta) = 8\mu_2^{-2}\eta(\mu_2^2 + \eta^2)^{-1/2} + \lambda_2$$
$$h_2(\eta) = 8\mu_2^{-2}(\mu_2^2 + \eta^2)^{1/2} + \lambda_2\eta$$

where λ_2 is a further arbitrary constant, and the transformations (3.29) and (3.30) map the solution into another solution of the same form.

(3) From (3.12)–(3.15) with $\nu_3=0$, using (3.29) gives

$$\begin{aligned} W &= h_3(\eta) \\ V &= g_3(\eta)/(T^{3/2}-\alpha g_3(\eta)) \\ U &= T^{3/2}f_3(\eta)/(T^{3/2}-\alpha g_3(\eta))^3 \end{aligned} \tag{3.33}$$

where $\eta=(X+\alpha W)/T^{3/2}$.

The transformation (3.30) leads to the separable solution.

(4) Under (3.29) the solutions (3.16)–(3.18) become

$$\begin{aligned} W &= T^{3/2}h_4(\eta) \\ V &= T^{3/2}g_4(\eta)/(1-\alpha T^{3/2}g_4(\eta)) \\ U &= T^{3/2}f_4(\eta)/(1-\alpha T^{3/2}g_4(\eta))^3 \end{aligned} \tag{3.34}$$

where $\eta=X+\alpha W$.

The transformation (3.30) gives a dipole solution. To within translations of W by a multiple of X, etc, the two forms (3.33) and (3.34) are equivalent.

(5) Under (3.29), solutions (3.20)–(3.23) become

$$\begin{aligned} W &= \mu_5\eta T+\nu_5 T+\frac{(\nu_5+\mu_5\eta)^5}{540\mu_5^2} \\ V &= \left(\mu_5 T+\frac{(\nu_5+\mu_5\eta)^4}{108\mu_5}\right)\Bigg/\left(1-\alpha\left(\mu_5 T+\frac{(\nu_5+\mu_5\eta)^4}{108\mu_5}\right)\right) \\ U &= (\nu_5+\mu_5\eta)^3\Bigg/27\left(1-\alpha\left(\mu_5 T+\frac{(\nu_5+\mu_5\eta)^4}{108\mu_5}\right)\right)^3 \end{aligned} \tag{3.35}$$

where $\eta=X+\alpha W$.

In this case (3.30) gives

$$\begin{aligned} X &= \mu_5 WT+\nu_5 T+\frac{(\nu_5+\mu_5 W)^5}{540\mu_5^2} \\ V &= 1\Bigg/\left(\mu_5 T+\frac{(\nu_5+\mu_5 W)^4}{108\mu_5}\right) \\ U &= -(\nu_5+\mu_5 W)^3\Bigg/27\left(\mu_5 T+\frac{(\nu_5+\mu_5 W)^4}{108\mu_5}\right)^3. \end{aligned} \tag{3.36}$$

When $\nu_5=0$, (3.36) may be written as a similarity solution of a standard form:

$$\begin{aligned} W &= T^{1/4}H(X/T^{5/4}) \\ V &= T^{-1}G(X/T^{5/4}) \\ U &= T^{-9/4}F(X/T^{5/4}) \end{aligned}$$

where

$$F(\eta)=\frac{\mathrm{d}G}{\mathrm{d}\eta}(\eta) \qquad G(\eta)=\frac{\mathrm{d}H}{\mathrm{d}\eta}(\eta)$$

and

$$\eta=\mu_5 H(\eta)+\frac{\mu_5^3 H^5(\eta)}{540}.$$

4. Some boundary value problems

4.1. Introduction

In this section we shall use the infinitesimal representations (2.20) and (2.21) to derive self-similar forms of solution for some particular boundary value problems. The appropriate similarity variables are determined by the requirement that the boundary conditions, as well as the differential equations, be invariant under the continuous transformations represented by the infinitesimals. We shall not be able to specify initial conditions in advance; these are restricted by the forms of the similarity solutions.

Knight and Philip [13] have commented on the paucity of exact solutions to nonlinear diffusion equations on finite regions, though they fail to note the separable solution (3.16)-(3.19) which is known to be important in describing the large-time behaviour of certain problems on bounded intervals (see [14, 15]). We shall start by deriving some new solutions on finite domains, after which we shall consider infinite regions.

4.2. Finite domains

4.2.1. $m=\frac{4}{3}$. We consider boundary value problems for (2.1) on the interval $0\leqslant x\leqslant L$. Although the infinitesimal $\hat{x}$ seems well suited to such problems, the corresponding similarity forms for u seem likely to have very restricted physical applicability. We require $\hat{x}=0$ at $x=0$ and at $x=L$, so it immediately follows from (2.20) that

$$a_2=0 \qquad a_3=a_7L. \tag{4.1}$$

Here we shall consider only one example, in which (2.1) is governed by

$$\text{at } x=0 \quad u=B_0t^k \qquad \text{at } x=L \quad u=B_Lt^p \qquad \text{for } t>0 \tag{4.2}$$

where B_0, B_L, k and p are constants. Replacing x by $L-x$ gives another problem of the same form (with the roles of B_0 and B_L and of k and p interchanged), so we may restrict attention to the case $k\leqslant p$.

We note that the choices $B_0=0$ and $B_L=0$ are not admissible. A quasi-steady local balance on (1.1) indicates the possibility of solutions with local behaviour

$$u\sim u_0(t)x^{1/(1-m)} \qquad \text{as } x\to 0 \tag{4.3}$$

or

$$u\sim u_L(t)(L-x)^{1/(1-m)} \qquad \text{as } x\to L \tag{4.4}$$

for some functions of u_0 and u_L; when these expressions describe the local behaviour then the flux $-u^{-m}u_x$ across $x=0$ or $x=L$ is finite. Since $m>1$ here, $u=0$ cannot be imposed on $x=0$ or $x=L$. The other limiting case, $u\to+\infty$ as $x\to 0$ or $x\to L$, is permissible, however, and corresponds to the limits $B_0\to+\infty$ or $B_L\to+\infty$ in (4.2).

Since the boundary conditions must be invariant under the transformation group we require that

$$\hat{u}=\hat{t}\frac{\partial u}{\partial t} \qquad \text{at } x=0 \text{ and at } x=L \tag{4.5}$$

implying that

$$
\begin{aligned}
&B_0 a_1 = 0 \qquad B_0 a_6 = B_0 k(2a_3 + \tfrac{4}{3}a_6) \\
&B_L a_1 = 0 \qquad B_L(a_6 + 3a_7 L) = B_L p(2a_3 + \tfrac{4}{3}a_6).
\end{aligned}
\tag{4.6}
$$

Unless B_0 or $B_L \to \infty$, then it follows from (4.1) and (4.6) that if $a_6 \neq 0$ and $k \neq \frac{3}{4}$ we need

$$
a_1 = 0 \qquad a_6 = 6ka_7 L/(3-4k) \qquad p + k = \tfrac{3}{2}
\tag{4.7}
$$

so that k and p cannot be chosen independently, and if $a_6 \neq 0$ and $k = \frac{3}{4}$ we need

$$
a_1 = 0 \qquad a_3 = 0 \qquad a_7 = 0 \qquad p = \tfrac{3}{4}
$$

which give the usual separable solution. Restricting attention to $k \leqslant p$ now implies that $k \leqslant \frac{3}{4}$.

For $k \neq \frac{3}{4}$ we may without loss of generality take $a_7 = 1$ in (4.1) and (4.7) to give infinitesimals

$$
\hat{u} = \left(\frac{6kL}{3-4k} + 3x\right)u \qquad \hat{x} = x(L-x) \qquad \hat{t} = \frac{6L}{3-4k}\,t.
$$

The corresponding similarity variables may be derived in the usual way and we may write the solution in the form

$$
u = t^k\left(1 - \frac{x}{L}\right)^{-3} f\left(x \Big/ \left(1 - \frac{x}{L}\right) t^{(3-4k)/6}\right).
\tag{4.8}
$$

From (2.25) it follows that (4.8) may be derived from the solution

$$
u = t^k f(\eta) \qquad \eta = x/t^{(3-4k)/6}
$$

where f is the same function as in (4.8). $f(\eta)$ then satisfies

$$
kf - \frac{3-4k}{6}\eta\frac{\mathrm{d}f}{\mathrm{d}\eta} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{-4/3}\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)
\tag{4.9}
$$

subject to

$$
\begin{aligned}
&\text{at } \eta = 0 \qquad f = B_0 \\
&\text{as } \eta \to +\infty \qquad f \sim B_L L^3 \eta^{-3}.
\end{aligned}
\tag{4.10}
$$

We will not attempt to discuss the parameter ranges under which there exists a solution to this problem.

When $k = -\frac{3}{2}$ we may integrate (4.9) once to give

$$
-\tfrac{3}{2}\eta f = f^{-4/3}\frac{\mathrm{d}f}{\mathrm{d}\eta} + 3B_L^{-1/3}L^{-1}.
$$

In the limit $B_L \to +\infty$ we then obtain

$$
f = (B_0^{-4/3} + \eta^2)^{-3/4}
$$

so that (4.8) becomes

$$
u = t^{-3/2}\left(1 - \frac{x}{L}\right)^{-3}\left(B_0^{-4/3} + x^2 \Big/ \left(1 - \frac{x}{L}\right)^2 t^3\right)^{-3/4}
$$

which is equivalent to a closed form solution given by Hill [12].

We note that the similarity solution (4.8), subject to (4.10) with B_L finite, corresponds to initial conditions

$$\text{at } t=0 \qquad x>0 \qquad u=0$$

and for $k<\frac{3}{4}$ the large-time behaviour takes the form

$$\text{as } t\to+\infty \qquad x<L \qquad u\sim B_0 t^k\left(1-\frac{x}{L}\right)^{-3}.$$

4.2.2. $m=\frac{2}{3}$. We now generate solutions to (2.4)–(2.6) on the domain $0\leq X\leq L$. We start by constructing solutions to (2.6) and we then derive the corresponding solutions to (2.5) and (2.4). Since we require $\hat{X}=0$ on $X=0$ and on $X=L$, it follows from (2.21) that for $A_7\neq 0$ we need

$$W=-A_2/A_7 \qquad \text{on } X=0$$

$$W=-(A_2+A_3L)/A_7 \qquad \text{on } X=L.$$

Replacing W by $W+(A_2+A_3X)/A_7$ leaves (2.6) unchanged, so we may without loss of generality consider boundary conditions

$$W=0 \qquad \text{on } X=0 \text{ and } X=L. \tag{4.11}$$

We then need $\hat{X}=\hat{W}=0$ at $X=0$, $W=0$ and at $X=L$, $W=0$, so it follows from (2.21) that

$$A_2=A_3=A_4=A_5=0$$

and the resulting infinitesimals are then

$$\hat{W}=A_6W \qquad \hat{X}=A_7W \qquad \hat{T}=A_1+\tfrac{2}{3}A_6T.$$

If $A_6=0$ we write $A_7/A_1=\gamma$ and the relevant similarity form is

$$W=h(X-\gamma WT).$$

The general solution of this type may be written

$$W^5+k_1W+k_0=540\gamma^{-3}(X-\gamma WT)$$

where k_0 and k_1 are arbitrary constants. It is not then possible to satisfy both boundary conditions (4.11).

If $A_6\neq 0$ we write $A_7/A_6=\gamma$, $A_1/A_6=-\frac{2}{3}T_0$ to give self-similar forms

$$W=-(T_0-T)^{3/2}h(X-\gamma W) \qquad T\leq T_0. \tag{4.12}$$

The choice $(T_0-T)^{3/2}$, rather than $(T-T_0)^{3/2}$, is made so that the boundary conditions may be satisfied for a real function h; thus the solution (4.12) extinguishes at a finite time $T=T_0$. We note that $W<0$ is needed to give $U>0$ and this motivates the choice of sign in (4.12). It follows from (2.23) that the solution (4.12) can be deduced from the separable solution

$$W=-(T_0-T)^{3/2}h(X)$$

which corresponds to $\gamma=0$.

Introducing $\eta = X - \gamma W$, we need $h(\eta)=0$ on $\eta=0$ and $\eta=L$ so that $h(\eta)$ is given by

$$\int_0^{Lh/8\alpha} \frac{\mathrm{d}s}{\sqrt{1-s^4}} = \begin{cases} (2\alpha/L)\eta & \eta < L/2 \\ (2\alpha/L)(L-\eta) & \eta > L/2 \end{cases} \tag{4.13}$$

where

$$\alpha \equiv \int_0^1 \frac{\mathrm{d}s}{\sqrt{1-s^4}}.$$

At $\eta = L/2$ we have $h = 8\alpha/L$, which is its greatest value. Equations (4.12) and (4.13) give an implicit expression for W.

It then follows that V is given by

$$V = \begin{cases} -1/(4(T_0-T)^{-3/2}((8\alpha/L)^4 - h^4(\eta))^{-1/2} + \gamma) & \eta < L/2 \\ 1/(4(T_0-T)^{-3/2}((8\alpha/L)^4 - h^4(\eta))^{-1/2} - \gamma) & \eta > L/2. \end{cases} \tag{4.14}$$

For V to be bounded we require that

$$T_0 - T < \left(\frac{L^2}{16\alpha^2|\gamma|}\right)^{2/3}$$

and this is also the condition that (4.12) and (4.13) give W as a single-valued function of X. Hence this solution is only valid sufficiently close to the extinction time. We have in particular that

$$V = -\left(\frac{L^2}{16\alpha^2}(T_0-T)^{3/2} + \gamma\right)^{-1} \qquad \text{and} \qquad \frac{\partial V}{\partial X} = 0 \qquad \text{at } X = 0$$

with

$$V = \left(\frac{L^2}{16\alpha^2}(T_0-T)^{-3/2} - \gamma\right)^{-1} \qquad \text{and} \qquad \frac{\partial V}{\partial X} = 0 \qquad \text{at } X = L.$$

At

$$X = \frac{L}{2} + (T_0-T)^{3/2}\frac{8\alpha\gamma}{L}$$

we have

$$V = 0$$

and

$$W = -(T_0-T)^{3/2}\frac{8\alpha}{L}$$

which is its minimum value.

Finally, U may be determined from

$$U = \frac{1}{27}\left(\frac{\partial W}{\partial T}\right)^3$$

giving

$$U = \begin{cases} \dfrac{(T_0-T)^{3/2}h^3(\eta)}{8(1+\gamma(T_0-T)^{3/2}((8\alpha/L)^4 - h^4(\eta))^{1/2}/4)^3} & \eta < \dfrac{L}{2} \\[2ex] \dfrac{(T_0-T)^{3/2}h^3(\eta)}{8(1-\gamma(T_0-T)^{3/2}((8\alpha/L)^4 - h^4(\eta))^{1/2}/4)^3} & \eta > \dfrac{L}{2}. \end{cases} \tag{4.15}$$

The solution (4.15) satisfies $U=0$ on $X=0$ and on $X=L$. Such problems for (1.1) with $0<m<1$ are of physical importance and have been discussed in detail by Berryman (see, for example, [16]) who studied the way in which the separable solution $u=(t_0-t)^{1/m}f(x)$ is approached as $t_0-t\to 0^+$. The family of exact solutions (4.15) provides a test case for such analyses, and tends to the separable solution as $T_0-T\to 0^+$, as required.

4.3. *Infinite domains*

The new solutions we are able to derive for infinite domain problems are all for $m=\frac{2}{3}$. For the case $m=\frac{4}{3}$, the requirement that $x\to\infty$ be invariant under the transformation represented by (2.20) gives $a_7=0$ (if we write $y=1/x$ then it follows that $\hat{y}=a_7-a_3y-a_2y^2$). The discussion of this case on p 298 of Bluman and Cole [1] neglected to consider the behaviour of $x\to\infty$; it is also misleading in that (2.1) does have acceptable solutions corresponding to initial conditions with $u=0$.

The two boundary value problems we discuss give solutions which preserve total mass, firstly for (2.5) and then for (2.4).

4.3.1. Mass-preserving solutions to (2.5). The relevant boundary conditions on (2.5) are

$$\begin{aligned}&\text{as } X\to-\infty \qquad V\to 0\\ &\text{as } X\to+\infty \qquad V\to 0.\end{aligned} \tag{4.16}$$

We assume that the total mass

$$Q\equiv\int_{-\infty}^{\infty} V\,\mathrm{d}X \tag{4.17}$$

is initially finite; (2.5) and (4.16) then imply that it is independent of T.

Writing

$$W=\int_{-\infty}^{X} V\,\mathrm{d}X$$

gives

$$\begin{aligned}&\text{as } X\to-\infty \qquad W\to 0\\ &\text{as } X\to+\infty \qquad W\to Q\end{aligned} \tag{4.18}$$

as conditions on (2.6). We therefore need $\hat{W}\to 0$ as $X\to-\infty$, $W\to 0$ and as $X\to+\infty$, $W\to Q$, giving

$$A_4=0 \qquad A_5=0 \qquad A_6=-2A_3.$$

If $A_3=0$ then the boundary conditions (4.18) cannot be satisfied by the resulting self-similar solution. If $A_3\neq 0$ we write $A_7/A_3=-2X_0/Q$, where X_0 is an arbitrary constant; by translations of T and X we may without loss of generality and for later convenience then take $A_1=0$ and $A_2=A_3X_0$ to give

$$\hat{W}=0 \qquad \hat{X}=A_3(X-2X_0W/Q+X_0) \qquad \hat{T}=\tfrac{2}{3}A_3T.$$

The resulting similarity solution takes the form

$$W=h(\eta) \qquad \text{with } \eta=(X-2X_0W/Q+X_0)/T^{3/2} \tag{4.19}$$

so that the solution is given parametrically by

$$W = h(\eta) \qquad X = 2X_0 h(\eta)/Q - X_0 + \eta T^{3/2}.$$

We have

$$h(\eta) = Qa \int_{-\infty}^{\eta a|Q|/4} (1+s^4)^{-1/2}\, \mathrm{d}s$$

where

$$a = \left(\int_{-\infty}^{\infty} (1+s^4)^{-1/2}\, \mathrm{d}s \right)^{-1}.$$

Therefore

$$V = g(\eta)/(T^{3/2} + 2X_0 g(\eta)/Q) \tag{4.20}$$

where

$$g(\eta) = 4\,\mathrm{sgn}(Q)((4/aQ)^4 + \eta^4)^{-1/2}.$$

The maximum value of $|g(\eta)|$ is $a^2Q^2/4$, and this occurs at $\eta = 0$, where $W = Q/2$ and $X = 0$. Hence for $X_0 < 0$ it follows from (4.20) that we require

$$T > (-X_0 a^2 |Q|/2)^{2/3}$$

for the solution to be acceptable.

We also obtain

$$U = T^{3/2} f(\eta)/(T^{3/2} + 2X_0 g(\eta)/Q)^3 \tag{4.21}$$

where

$$f(\eta) = -8\,\mathrm{sgn}(Q)\eta^3((4/aQ)^4 + \eta^4)^{-3/2}.$$

We note that at $X = 0$

$$U = \frac{\partial V}{\partial X} = 0.$$

We also note that (4.17) implies that

$$\int_{-\infty}^{\infty} XU\, \mathrm{d}X = -Q.$$

If $X_0 > 0$ then we have the following initial conditions at $T = 0$:

$$W = \frac{Q}{2X_0}((X+X_0)H(X+X_0) - (X-X_0)H(X-X_0))$$

$$V = \frac{Q}{2X_0}(H(X+X_0) - H(X-X_0))$$

$$U = \frac{Q}{2X_0}(\delta(X+X_0) - \delta(X-X_0))$$

where $H(Y)$ is the Heaviside step function. We have thus in particular generated a solution to the initial-boundary value problem for (2.4) subject to

at $T = 0$	$U = M\delta(X - X_0)$
at $X = 0$	$U = 0$
as $X \to +\infty$	$U \to 0$

where $M = -Q/2X_0$ is the total mass at $T = 0$. As might be expected, as $T \to 0$ this solution behaves asymptotically as an instantaneous source solution for U centred at $X = X_0$, while for large T it approaches the usual dipole solution. The solution satisfies

$$\int_0^\infty XU \, \mathrm{d}X = MX_0$$

for all T.

The solution for V generalizes the solution given in section 4.12 of Dresner [7] which corresponds to setting $X_0 = 0$.

4.3.2. Mass-preserving solutions to (2.4). We now seek solutions to (2.4) such that the total mass

$$M \equiv \int_{-\infty}^\infty U \, \mathrm{d}X$$

which is assumed finite, is independent of T. The relevant boundary conditions are

$$\begin{aligned} &\text{as } X \to +\infty \qquad U \to 0 \\ &\text{as } X \to +\infty \qquad U \to 0. \end{aligned} \tag{4.22}$$

Mass-preserving solutions to (2.4) are of interest in, for example, semiconductor applications [8].

Writing

$$V = \int_{-\infty}^X U \, \mathrm{d}X$$

then gives

$$\begin{aligned} &\text{as } X \to -\infty \qquad V \to 0 \\ &\text{as } X \to +\infty \qquad V \to M \end{aligned} \tag{4.23}$$

as conditions on (2.5).

We now write

$$W = \int_{-\infty}^X V \, \mathrm{d}X \tag{4.24}$$

to give

$$\begin{aligned} &\text{as } X \to -\infty \qquad W \to 0 \\ &\text{as } X \to +\infty \qquad W = MX + o(X). \end{aligned} \tag{4.25}$$

We note that if the first moment

$$\int_{-\infty}^\infty XU \, \mathrm{d}X \tag{4.26}$$

is unbounded then the integral (4.24) may not exist. However, the similarity solutions we can derive turn out to have at least one of

$$\int_{-\infty}^X V \, \mathrm{d}X \qquad \text{and} \qquad \int_X^\infty (V - M) \, \mathrm{d}X$$

bounded. We consider the former case; solutions for the latter case can then be obtained by writing

$$W^+ = W - MX \qquad X^+ = -X \qquad T^+ = T \tag{4.27}$$

with

$$U^+(X^+, T^+) = U(-X, T).$$

We note that even when (4.26) is unbounded, the integral result

$$\int_{-\infty}^{\infty} X(U - U_0)\,\mathrm{d}X = 0$$

holds, where $U = U_0(X)$ at $T = 0$.

The conditions (4.25) require that

$$A_4 = A_5 = 0 \qquad A_6 = MA_7 - A_3$$

so that

$$\begin{aligned} \hat{W} &= (A_3 + MA_7)W \\ \hat{X} &= A_2 + A_3X + A_7W \\ \hat{T} &= A_1 + \tfrac{2}{3}(2A_3 + MA_7)T \end{aligned} \tag{4.28}$$

giving

$$\hat{X} - \hat{W}/M = A_2 + A_3(X - W/M).$$

Before proceeding further it is instructive to consider the instantaneous source case when

$$\text{at } T = 0 \qquad U = M\delta(X) \qquad V = MH(X) \qquad W = MXH(X).$$

These initial conditions require that $A_1 = A_2 = 0$, leaving

$$\begin{aligned} \hat{W} &= (A_3 + MA_7)W \\ \hat{X} &= A_3X + A_7W \\ \hat{T} &= \tfrac{2}{3}(2A_3 + MA_7)T \end{aligned} \tag{4.29}$$

so that the problem is invariant under a two-parameter group. This means that the functional form of the solution may be determined without further reference to the differential equation; the analogous problem in linear diffusion is discussed in Bluman and Cole [1] pp 221–6. The similarity form corresponding to (4.29) may be written

$$X = \frac{W}{M} + T^{\gamma}h(W/T^{3/2-\gamma}) \tag{4.30}$$

where $\gamma = 3A_3/2(2A_3 + MA_7)$. Since the instantaneous source solution must take the form (4.30) for all values of γ, we need

$$h(\eta) = k/\eta$$

where k is a constant, giving

$$X = \frac{W}{M} + \frac{kT^{3/2}}{W}.$$

To determine k we must substitute into (2.6), giving $k^2 = 16$. Assuming $M > 0$ we need $k = -4$ so that

$$X = \frac{W}{M} - \frac{4T^{3/2}}{W} \tag{4.31}$$

and

$$W = (MX + (M^2X^2 + 16MT^{3/2})^{1/2})/2.$$

Equation (4.31) describes the large time behaviour of the solutions we shall now derive.

Returning to (4.28) we must consider three subcases as follows.

(1) $A_3 \neq 0, 2A_3 + MA_7 \neq 0$. By translations of T and X we may then without loss of generality take

$$A_1 = A_2 = 0$$

so that we recover (4.29) and the similarity solutions take the form (4.30). Initial conditions appropriate to the self-similar forms (4.30) are

$$\text{at } T = 0 \qquad X = \frac{W}{M} - KW^{\mu} \tag{4.32}$$

where $\mu = 2\gamma/(3 - 2\gamma)$ and K is an arbitrary constant. We note that when $K \neq 0$ we could without loss of generality take $M = 1$, $|K| = 1$ by replacing

$$W \text{ by } (M|K|)^{1/(1-\mu)} W$$

$$X \text{ by } (M^{\mu}|K|)^{1/(1-\mu)} X$$

and

$$T \text{ by } (M^{1+\mu}K^2)^{2/(3(1-\mu))} T.$$

When $K = 0$ the solution is given by (4.31) for all γ. If $K \neq 0$ then under the transformation (4.27) the initial condition (4.32) becomes

$$\text{at } T^+ = 0 \qquad X^+ = \frac{W^+}{M} - K^+ W^{+1/\mu}$$

where $K^+ = (MK)^{-1/\mu}/M$. We may therefore without loss of generality restrict attention to the range $\gamma < \frac{3}{4}$ giving $-1 < \mu < 1$. We note that $\mu = 1$ is not acceptable for $K \neq 0$ because we require $W \sim MX$ as $X \to +\infty$. In (4.30) the function $h(\eta)$ satisfies

$$\begin{aligned} &\gamma h + (\gamma - \tfrac{3}{2})\eta \frac{dh}{d\eta} = 3\left(\frac{d^2h}{d\eta^2}\right)^{1/3} \\ &\text{as } \eta \to 0^+ \qquad h \to -\infty \\ &\text{as } \eta \to +\infty \qquad h \sim -K\eta^{\mu}. \end{aligned} \tag{4.33}$$

An asymptotic balance shows that, more precisely,

$$\text{as } \eta \to 0^+ \qquad h \sim -4/\eta \tag{4.34}$$

so that the large-time behaviour is given by (4.31).

We shall need to discuss the cases $\mu<0$ and $\mu>0$ separately. We first note the following borderline cases:

(a) $\mu=0$ ($\gamma=0$). Solving (4.33) then gives

$$h=-K-4/\eta$$

which gives the instantaneous source solution (4.31) with X translated by K.

(b) $\mu=1$ ($\gamma=\frac{3}{4}$). The solution to (4.33) is then

$$h=-K\eta-4/\eta$$

giving the instantaneous source solution for total mass $M/(1-KM)$.

(c) $\mu=-1$ ($\gamma\to\infty$). Returning to the initial condition (4.32) it is clear that the solution to (2.6) is given by

$$X=\frac{W}{M}-\frac{4(T+(K/4)^{2/3})^{3/2}}{W} \tag{4.35}$$

which is the instantaneous source solution (4.31) with T translated by $(K/4)^{2/3}$.

We now discuss the remaining cases.

(i) $0<\mu<1$ ($0<\gamma<\frac{3}{4}$). Initial conditions are given by (4.32) for $X>0$ together with

$$\text{at } T=0 \qquad W=0 \qquad \text{for } X\leqslant 0.$$

For the solution to make sense for all $T>0$ we need $K<0$ and then

$$\text{at } T=0 \qquad W\sim(-X/K)^{1/\mu} \qquad \text{as } X\to 0^{+}.$$

For $K>0$ the solution is multivalued for sufficiently small T but is acceptable for

$$T>(M\sup(-g(\eta)))^{2/(3-4\gamma)} \tag{4.36}$$

where $g=\mathrm{d}h/\mathrm{d}\eta$. When $K<0$ the condition $g>0$ holds.

(ii) $-1<\mu<0$ ($\gamma<0$). In this case (4.32) describes the initial conditions for all X. If $K>0$ the solution is valid for all $T>0$ and

$$\text{at } T=0 \qquad W\sim(-X/K)^{1/\mu} \qquad \text{as } X\to-\infty.$$

For $K<0$ the solution is acceptable only when the condition (4.36) is met.

For both (i) and (ii) we have

$$\text{as } X\to+\infty \qquad W\sim MX+KM^{1+\mu}X^{\mu}$$

so that the first moment (4.26) is unbounded if $\mu>0$ and

$$\text{as } X\to-\infty \qquad W\sim-4T^{3/2}/X \qquad \text{for } T>0.$$

The solutions for V and U are given by

$$V=M/(1+MT^{2\gamma-3/2}g(\eta))$$
$$U=-M^3T^{3\gamma-3}f(\eta)/(1+MT^{2\gamma-3/2}g(\eta))^3$$

where

$$\eta=W/T^{3/2-\gamma} \qquad f=\frac{\mathrm{d}g}{\mathrm{d}\eta}=\frac{\mathrm{d}^2h}{\mathrm{d}\eta^2}.$$

When $K>0$ in case (i) and $K<0$ in case (ii), U is negative for sufficiently large positive X.

(2) $A_3=0$, $A_7\neq 0$. By translating T we may again without loss of generality take $A_1=0$ to give

$$\hat{W}=MA_7W \qquad \hat{X}-\hat{W}/M=A_2 \qquad \hat{T}=\tfrac{2}{3}MA_7T$$

and the resulting similarity solution takes the form

$$X=\frac{W}{M}+\tfrac{3}{2}\nu\ln T+h(W/T^{3/2}) \tag{4.37}$$

where $\nu=A_2/MA_7$. The similarity ordinary differential equation is then

$$\tfrac{1}{2}\left(\nu-\eta\frac{\mathrm{d}h}{\mathrm{d}\eta}\right)=\left(\frac{\mathrm{d}^2h}{\mathrm{d}\eta^2}\right)^{1/3}. \tag{4.38}$$

Writing $g=\mathrm{d}h/\mathrm{d}\eta$ gives

$$\tfrac{1}{2}(\nu-\eta g)=\left(\frac{\mathrm{d}g}{\mathrm{d}\eta}\right)^{1/3}$$

which is to be solved subject to $g\to+\infty$ as $\eta\to 0^+$; this implies that $g\sim 4/\eta^2$ as $\eta\to 0^+$. The behaviour as $\eta\to+\infty$ is given by $g\sim\nu/\eta$.

The arbitrary constant which arises on integrating g to give h corresponds to a translation of X. Writing

$$h=\nu\ln\eta-\int_\eta^\infty(g-\nu/\eta)\,\mathrm{d}\eta$$

we have $h\sim -4/\eta$ as $\eta\to 0^+$ and $h=\nu\ln\eta+o(1)$ as $\eta\to+\infty$.

If $\nu>0$ the solution is valid for all $T>0$ and we have

$$\text{at } T=0 \qquad X=\frac{W}{M}+\nu\ln W$$

with

as $X\to+\infty$ $\qquad W\sim MX-\nu M\ln(MX)$ $\qquad$ for all T

as $X\to-\infty$ $\qquad W\sim\exp(X/\nu)$ at $T=0$ $\qquad W\sim -4T^{3/2}/X$ $\qquad$ for $T>0$.

If $\nu<0$ the solution is only valid if

$$T>(M\sup(-g(\eta)))^{2/3}.$$

The large time behaviour is given by (4.31).

(3) $A_3\neq 0$, $2A_3+MA_7=0$. By translating X we may take $A_2=0$ giving similarity solutions

$$X=\frac{W}{M}+\mathrm{e}^{-\lambda T}h(W/\mathrm{e}^{\lambda T})$$

where $\lambda=-A_3/A_1$. The similarity ordinary differential equation for $h(\eta)$ is

$$-\lambda\left(h+\eta\frac{\mathrm{d}h}{\mathrm{d}\eta}\right)=3\left(\frac{\mathrm{d}^2h}{\mathrm{d}\eta^2}\right)^{1/3}. \tag{4.39}$$

We require $h \to -\infty$ as $\eta \to 0^+$, and (4.39) then implies that

$$\text{as } \eta \to 0^+ \qquad h \sim -\frac{4}{\lambda^{3/2}\eta} \ln^{3/2}\left(\frac{1}{\eta}\right)$$

requiring $\lambda > 0$, whereas a far-field balance indicates that

$$\text{as } \eta \to +\infty \qquad h \sim \pm\frac{4}{(-\lambda)^{3/2}\eta} \ln^{3/2}\eta.$$

It therefore seems that there are no solutions of this form satisfying the required boundary conditions.

In this subsection we have in cases (1) and (2) constructed new mass-preserving solutions for U which may be determined from similarity forms for W. Closed form solutions of the relevant similarity ordinary differential equations cannot in general be determined, however.

5. Discussion

We begin by considering a generalization of (1.1), namely

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(K(u)) \tag{5.1}$$

together with its integrated forms

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(K\left(\frac{\partial v}{\partial x}\right)\right) \tag{5.2}$$

and

$$\frac{\partial w}{\partial t} = K\left(\frac{\partial^2 w}{\partial x^2}\right) \tag{5.3}$$

where

$$u = \frac{\partial v}{\partial x} \qquad v = \frac{\partial w}{\partial x}.$$

We now write

$$\begin{aligned} &u = (a_1 u^\dagger + b_1)/(a_2 u^\dagger + b_2^\dagger) \qquad v = a_1 v^\dagger + b_1 x^\dagger \\ &w - \tfrac{1}{2} x v = w^\dagger - \tfrac{1}{2} x^\dagger v^\dagger \\ &x = c v^\dagger + \mathrm{d} x^\dagger \qquad t = t^\dagger \end{aligned} \tag{5.4}$$

which generalizes transformations we have already considered; a_1, b_1, a_2 and b_2 are constants such that $a_1 b_2 - b_1 a_2 = 1$.

Relations (5.4) imply that

$$u^\dagger = \frac{\partial v^\dagger}{\partial x^\dagger} \qquad v^\dagger = \frac{\partial w^\dagger}{\partial x^\dagger}$$

with

$$\frac{\partial u^\dagger}{\partial t^\dagger} = \frac{\partial^2}{\partial x^{\dagger 2}}(K^\dagger(u^\dagger)) \tag{5.5}$$

where

$$K^{\dagger}(u^{\dagger}) = K\left(\frac{a_1 u^{\dagger} + b_1}{a_2 u^{\dagger} + b_2}\right) \tag{5.6}$$

so that equation (5.1) may be mapped into the more general equation (5.5).

If we consider the related transformation

$$\begin{aligned} v &= (A_1 v^+ + B_1)/(A_2 v^+ + B_2) \qquad & w &= A_1 w^+ + B_1 x^+ \\ x &= A_2 w^+ + B_2 x^+ & t &= t^+ \end{aligned} \tag{5.7}$$

where A_1, B_1, A_2 and B_2 are constants such that $A_1 B_2 - B_1 A_2 = 1$, then we have $v^+ = \partial w^+/\partial x^+$, and equation (5.2) is mapped to

$$\frac{\partial v^+}{\partial t^+} = \frac{\partial}{\partial x^+}\left((A_2 v^+ + B_2) K\left(\frac{\partial v^+}{\partial x^+}\frac{1}{(A_2 v^+ + B_2)^3}\right)\right).$$

Combining the two transformations (5.4) and (5.7), equation (5.2) is mapped to

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left((A_2 v + B_2) K\left(\frac{a_1(\partial v/\partial x) + b_1(A_2 v + B_2)^3}{a_2(\partial v/\partial x) + b_2(A_2 v + B_2)^3}\right)\right)$$

by appropriate redefinitions of the variables. It follows in particular that the equation

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left((A_2 v + B_2)\left(\frac{a_1(\partial v/\partial x) + b_1(A_2 v + B_2)^3}{a_2(\partial v/\partial x) + b_2(A_2 v + B_2)^3}\right)\right)$$

is exactly linearizable.

Returning to equations (5.1) and (5.5), it follows from (5.6) that equations (2.1) and (2.4) may both be mapped to

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2}{\partial x^2}\left(\left(\frac{\alpha u + \beta}{\gamma u + \delta}\right)^{1/3}\right) \tag{5.8}$$

where $\alpha\delta - \beta\gamma = 1$, again by suitable redefinitions of the variables. The symmetry group of (5.8) arising from the symmetries (2.22) and (2.23) of (2.1) and (2.4) is not a point transformation at any of the three levels (5.1) to (5.3), and equation (5.8) was not noted as a special case by Bluman *et al* [17]. The relevant transformation is, however, a point transformation for the system

$$v = \frac{\partial w}{\partial x} \qquad \frac{\partial w}{\partial t} = K\left(\frac{\partial v}{\partial x}\right)$$

which implies both (5.2) and (5.3). Indeed all the symmetries we have discussed for equations of the form (5.1) to (5.3) arise as point transformations of this system. Bluman *et al* [17] considered the system of the form

$$u = \frac{\partial v}{\partial x} \qquad \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(K(u))$$

which implies both (5.1) and (5.2). Some of the transformations we have considered are not local symmetries for this system (see (2.22) and Bluman and Kumei [18] p 362).

Using the transformations of sections 1 and 2 of King [19], we may use the solutions of section 4.3.2 to generate new mass-preserving solutions to

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2/3}\left(1 + \frac{u}{u_0}\right)^{-4/3}\frac{\partial u}{\partial x}\right) \tag{5.9}$$

which is a special case of (5.8), u_0 being a constant. Such solutions with mass M may also be derived (as in section 3 of King [19]) from solutions to the boundary-value problem

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right)$$

$$\text{as } x\to 0 \qquad u\to +\infty$$

$$\text{as } x\to M \qquad u\to +\infty.$$

These boundary conditions may be applied to solutions of the form (4.8) with $L=M$, and these map to solutions in section 4.3.2. Such solutions did not arise in the analysis of King [19] which considered only even mass preserving solutions; the resulting solutions to (5.9) are not even in x.

We now make some comments about our similarity solutions. Firstly, it is evident that many of these solutions, such as those of subsection 4.3.1, satisfy initial conditions which contain a lengthscale. Usually this would make solution by similarity methods impossible; the additional parameter in the symmetry groups we have been discussing makes it feasible in these special cases.

Similarity forms such as (4.30) are also unusual in that the similarity variable depends on the dependent variable W. This has implications for the direct method of Clarkson and Kruskal [20] which is intended to lead to all classical similarity forms, as well as to non-classical ones. Their assumption that the similarity variable depends only on the independent variables implies that their method will not determine similarity forms such as (4.30); in other words the assertion immediately following equation (1.3) of [20] does not seem to be correct. If we consider a partial differential equation for $w(x,t)$ then the invariant surface condition

$$\hat{x}\frac{\partial w}{\partial x}+\hat{t}\frac{\partial w}{\partial t}=\hat{w} \tag{5.10}$$

used to determine possible similarity forms (see, for example, Bluman and Cole [1]) has two invariants which are constant on each characteristic. We write these as

$$I_1\equiv I_1(x,t,w) \qquad I_2\equiv I_2(x,t,w).$$

(Note that we can write

$$I_1=(X-W/M)/T^{\gamma} \qquad I_2=W/T^{3/2-\gamma}$$

in the case of (4.30).)

A similarity solution can then be written in the form

$$I_1=h(I_2).$$

If there exists a function F such that

$$\frac{\partial}{\partial w}F(I_1,I_2)=0 \tag{5.11}$$

then we may take F as the similarity variable, and the similarity solution is then of the form discussed by Clarkson and Kruskal [20]. It is easily seen that (5.11) implies that $\hat{x}/\hat{t}$ is independent of w, or that $\hat{t}=0$. In practice this is very often the case but it need not always be so, as we have illustrated.

We conclude by summarizing some of the main features of our results. Firstly, the results for $m = \frac{2}{3}$ (sections 3.3, 4.2.2 and 4.3) are of particular significance because they are based on non-local symmetries of the type introduced in [10] and [17]. The problems discussed in sections 4.2.2 and 4.3 are among the first boundary value problems to be solved by such techniques.

The physical relevance of our solutions may be illustrated by the following applications. Equation (1.1) subject to conditions

$$u = 0 \qquad \text{at } x = 0 \text{ and at } x = L$$

arises in a variety of applications, including plasma physics (see [16]); the solution (4.15) applies to this case. The solution (4.20) can be used to represent a heat pulse in superfluid helium, generalizing the solution to (2.5) given in section 4.12 of [7] to the initial conditions

$$\text{at } T = 0 \qquad V = \begin{cases} \dfrac{Q}{2X_0} & \text{for } |X| < X_0 \\ 0 & \text{for } |X| > X_0 \end{cases}$$

rather than the delta function initial conditions of [7]. The corresponding solution to (2.4) given by (4.21) is relevant to problems on a semi-infinite domain in which the surface acts as a perfect sink so that

$$\text{at } X = 0 \qquad U = 0.$$

Finally, in section 4.3.2 we constructed a family of mass-preserving solutions for which (2.4) may be reduced to a similarity ordinary differential equation. Mass-preserving solutions have a wide range of possible applications, and the new solutions given here again generalize existing solutions which correspond to delta function initial conditions.

Many of the results of this paper can be extended to significantly more general classes of nonlinear diffusion equation, and such generalizations will be reported elsewhere.

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